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Ground state solutions for a self-interacting scalar field confined in a cavity

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Abstract
We look at the ground state, or vacuum, solutions for self-interacting scalar fields confined in cavities where the boundary conditions can rule out constant field configurations, other than the zero field. If the zero field is unstable, symmetry breaking can occur to a non-constant field configuration that has a lower energy. The stability of the zero field is determined by the size of the length scales that characterize the cavity and parameters that enter the scalar field potential. There can be a critical length at which an instability of the zero field sets in. In addition to looking at the rectangular and spherical cavity in detail, we describe a general method which can be used to find approximate analytical solutions when the length scales of the cavity are close to the critical value.

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1. Introduction

The idea of spontaneous symmetry breaking plays a central role in our current understanding of the standard model. (For an excellent treatment by one of the major contributors see reference [1] for example.) A toy model potential demonstrating the most basic feature of spontaneous symmetry is the double-well potential for a real scalar field considered by Goldstone [2]. (See (2.2) below.) This potential has minimum away from $\varphi = 0$ at non-zero values of $\varphi = \pm a$, and it is easy to show that these non-zero values represent possible stable ground states for the theory. We will only be concerned with classical scalar fields in this paper.

For flat Minkowski spacetime, the homogeneity and isotropy of the spacetime imply that the ground state must be constant. However, it is easy to envisage a situation in which this is no longer the case, even for homogeneous and isotropic spacetimes. A simple example was provided by Avis and Isham [3] where the spatial section of the spacetime was a torus. In this case, since a real scalar field is regarded as a cross-section of the real-line bundle with the spacetime as the base space, there is the possibility of a twisted field. An equivalent observation is that we could choose antiperiodic boundary conditions for the field on the torus just as well as periodic boundary conditions. Antiperiodic boundary conditions mean that the only constant field solution consistent with the boundary conditions is the zero field. For the double-well potential, this raises the interesting question of what the ground state is if cannot correspond to a minimum of the potential. What Avis and Isham [3] showed was that if the torus was below a critical size set by the parameters of the double-well potential, $\varphi = 0$ was the stable ground state; however, if the torus was larger than the critical size, $\varphi = 0$ was not stable, and the stable ground state was a spatially dependent solution. Thus, the boundary conditions can prohibit constant, non-zero field solutions, and make the determination of the ground state more complicated.

The main purpose of the present paper is to show that the situation for scalar fields confined in cavities is similar to that studied by Avis and Isham [3] for twisted scalar fields. In section 2, we will present a fairly general analysis showing how to characterize the critical size of cavities corresponding to the stability (or instability) of the zero field as the ground state. In situations where $\varphi = 0$ is unstable, an exact determination of the ground state is not possible in general. However, when the length scales of the cavity are close to the critical values at which an instability of $\varphi = 0$ sets in, it is possible to find approximate analytical solutions; this is described in section 3. One case where an exact analytical solution is possible occurs for a rectangular cavity where the field vanishes on only two opposite pairs of the cavity walls. This case is studied in subsection 4.1. In subsection 4.2, we use the approximate method of section 3 to obtain a solution which is shown to agree with an expansion of the exact solution we found in subsection 4.1. In subsection 4.3, we study the case where the field is totally confined by the cavity, vanishing on all of the box walls. The case of a spherical cavity, with the field vanishing on the surface of a spherical shell, is examined in section 5. No exact solution could be found, but we obtain a numerical solution, as well as apply the approximate method of section 3. The final section contains a brief discussion of the results.

2. General framework

We will consider a general region of D-dimensional Euclidean space Σ with boundary $\partial\Sigma$. (The basic formalism, we will describe, does not require the space to be flat, but we will not consider curved space in this paper.) The action functional for a real scalar field φ will be chosen to be

$$S[\varphi] = \int \mathrm{d}t \int_\Sigma \mathrm{d}\sigma_x \left\{ \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - U(\varphi) \right\} \tag{2.1}$$

where $U(\varphi)$ is some potential, and $\mathrm{d}\sigma_x$ denotes the invariant volume element for Σ which of course depends on the choice of coordinates. We will concentrate on the case of the double-well potential

$$U(\varphi) = \frac{\lambda}{4!}(\varphi^2 - a^2)^2 \tag{2.2}$$

in this paper, although the analysis is easily extended to other potentials in a straightforward manner. The field equation which follows from (2.1) is

$$\Box\varphi + U'(\varphi) = 0. \tag{2.3}$$

The aim is to solve this for φ in the region Σ, subject to some boundary conditions imposed on φ on the boundary $\partial\Sigma$ of Σ.

The energy functional (which is just the spatial integral of the Hamiltonian density) is

$$E[\varphi] = \int_\Sigma \mathrm{d}\sigma_x \left\{ \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}|\nabla\varphi|^2 + U(\varphi) \right\}. \tag{2.4}$$

Because the spacetime is static, if we concentrate on the lowest energy solution, or ground state, we would expect it to be static as well, resulting in no contribution to the kinetic energy term in (2.4). For static solutions, (2.3) becomes

$$-\nabla^2\varphi + U'(\varphi) = 0. \tag{2.5}$$

If we assume that φ has no time dependence, then (2.4) reads

$$E[\varphi] = \int_\Sigma \mathrm{d}\sigma_x \left\{ \frac{1}{2}\varphi(-\nabla^2)\varphi + U(\varphi) \right\} \tag{2.6}$$

after an integration by parts, provided that the boundary conditions are such that $\varphi\nabla_n\varphi$ vanishes on $\partial\Sigma$, with ∇_n the outwards directed normal derivative. (i.e., $\nabla_n = \hat{n}\cdot\nabla$ with $\hat{n}$ a unit normal vector on $\partial\Sigma$ which is directed out of the boundary.) We will assume that this is the case here.

For the potential (2.2), it is obvious that $\varphi = 0$ and $\varphi = \pm a$ are solutions to (2.5). There are two key issues that arise at this stage. The first is whether or not the boundary conditions on $\partial\Sigma$ are satisfied; if not, then the solution must be rejected. The second is, even if the boundary conditions are obeyed, the solution must be stable to perturbations. Assuming that we have a solution φ to (2.5) which satisfies the boundary conditions on $\partial\Sigma$, we can look at the energy of a perturbed solution $\varphi + \psi$ where ψ is treated as small. From (2.6), we find

$$E[\varphi + \psi] - E[\varphi] = \frac{1}{2}\int \mathrm{d}\sigma_x \psi(-\nabla^2 + U''(\varphi))\psi \tag{2.7}$$

if we work to the lowest order in ψ. By considering the eigenvalue problem

$$(-\nabla^2 + U''(\varphi))\psi_n = \lambda_n\psi_n \tag{2.8}$$

it can be seen that if any of the eigenvalues λ_n are negative, then $E[\varphi + \psi_n] < E[\varphi]$ and we would conclude that the solution φ is unstable to small perturbations since a solution of lower energy exists. If all of the eigenvalues are positive, then φ is stable to small perturbations (locally stable).

Suppose that we consider the solution $\varphi = 0$ to (2.5) with the potential (2.2). The eigenvalue equation (2.8) becomes

$$\left(-\nabla^2 - \frac{\lambda}{6}a^2\right)\psi_n = \lambda_n\psi_n. \tag{2.9}$$

If we consider the lowest eigenvalue λ_0, whose sign determines the stability of $\varphi = 0$, this sign requires knowing the smallest eigenvalue of the Laplacian $-\nabla^2$. Let

$$-\nabla^2\psi_0 = \ell_0^2\psi_0 \tag{2.10}$$

where $\ell_0^2 \geqslant 0$ and is real. We then have

$$\lambda_0 = \ell_0^2 - \frac{\lambda}{6}a^2. \tag{2.11}$$

If the boundary conditions on φ (and hence ψ_n) are such that $\ell_0^2 = 0$, then we can conclude that $\lambda_0 < 0$ and so $\varphi = 0$ is unstable. This is the case where the boundary conditions allow constant values of the field, and is the case for flat Euclidean space. However, if the boundary conditions prohibit constant values of the field, we will have $\ell_0^2 > 0$, and the stability or instability of $\varphi = 0$ is determined by the magnitude of ℓ_0^2 in relation to $\frac{\lambda}{6}a^2$. Typically, ℓ_0^2 will depend on the length scales describing Σ and its boundary, and so we will find critical values of these lengths at which $\varphi = 0$ becomes unstable. The ground state will not be constant in this case. The first demonstration of this was given in [3] for the case where Σ was a circle, or a torus, with the field φ satisfying antiperiodic boundary conditions. We will see examples of this for fields in cavities later in this paper.

The other constant solutions to (2.5) are $\varphi = \pm a$. The eigenvalue equation (2.8) reads

$$\left(-\nabla^2 + \frac{\lambda}{3}a^2\right)\psi_n = \lambda_n \psi_n \tag{2.12}$$

for these solutions. In place of (2.11), we have

$$\lambda_0 = \ell_0^2 + \frac{\lambda}{3}a^2 \tag{2.13}$$

with ℓ_0^2 defined as in (2.10). Provided that the boundary conditions allow constant values of the field, if $\varphi = 0$ is unstable, we see that $\lambda_0 > 0$ implying stability of the solutions $\varphi = \pm a$. This is the usual situation in Euclidean space [2]. However, if the boundary conditions rule out $\varphi = \pm a$ as possible solutions, then the ground state when $\varphi = 0$ is unstable necessarily involves non-constant fields.

A simple example where constant field values (other than zero) are not allowed occurs for a field confined to a cavity Σ with Dirichlet boundary conditions imposed on the boundary $\partial\Sigma$ of the cavity. (i.e., $\varphi = 0$ on $\partial\Sigma$.) In this case, if $\varphi = 0$ is not stable, we are faced with the question of what the ground state is. In section 4, we will look at the simplest case of a cavity which is rectangular. In section 5, we will study what happens for a spherical cavity. The case of the rectangular box allows for an exact solution for the ground state for some boundary conditions; whereas for the spherical cavity, we were not able to solve for the ground state except numerically. Before proceeding to these examples, we will present a method for obtaining approximate analytical solutions for the ground state.

3. Approximate analytical results

In [4], a systematic method was described for obtaining approximate solutions for the ground state in cases where it was not possible to solve (2.5) exactly (except numerically). In this section, we will present a variation on this method applicable to general cavities. The method described in [4] was based on an earlier attempt by Banach [5] to define an effective potential which could be used for twisted fields. The relationship between Banach's [5] approach and that of Laura and Toms is discussed in [4].

We saw that the stability of $\varphi = 0$ relied on the magnitude of ℓ_0^2 which in turn depends on the length scales of the cavity. For simplicity, assume that there is only one length scale L in the problem. Let L_c be the critical value of this length defined by the condition

$$\ell_0^2 = \frac{\lambda}{6}a^2. \tag{3.1}$$

On dimensional grounds, we must have $\ell_0^2 \propto L^{-2}$, so for $L < L_c$ we expect $\lambda_0 > 0$ and hence for $\varphi = 0$ to be locally stable. When $L > L_c$, we have $\lambda_0 < 0$, and so $\varphi = 0$ becomes unstable. In this case, the ground state is given by a solution other than $\varphi = 0$.

Suppose that we take the length scale L to be slightly greater than the critical value:

$$L = (1+\epsilon)L_c \tag{3.2}$$

with $\epsilon > 0$ treated as small. We can scale the coordinates with appropriate factors of L to enable us to use dimensionless coordinates. The Laplacian $-\nabla^2$ can then be expanded in powers of ϵ using (3.2) to give

$$-\nabla^2 = -\nabla_c^2 - \epsilon\nabla_1 - \epsilon^2\nabla_2 - \cdots \tag{3.3}$$

where ∇_c^2 denotes that the Laplacian is evaluated with $L = L_c$, and $\nabla_1, \nabla_2, \ldots$ are differential operators whose form is determined by the Laplacian and the specific cavity. (We will apply this to the rectangular box and the spherical cavity later to show how this works in practice.)

An argument presented in [4], which is substantiated by application to specific examples, may be repeated to show that the solution to (2.5) can be written as

$$\varphi(x) = \epsilon^{1/2}[\varphi_0(x) + \epsilon\varphi_1(x) + \epsilon^2\varphi_2(x) + \cdots] \tag{3.4}$$

where the functions $\varphi_0(x), \varphi_1(x), \ldots$ appearing on the right-hand side are independent of ϵ. Using (3.3) and (3.4) in (2.5), with the potential given by (2.2), and then equating equal powers of ϵ to zero, we obtain a set of coupled differential equations, the first two of which are

$$-\nabla_c^2\varphi_0 - \frac{\lambda}{6}a^2\varphi_0 = 0 \tag{3.5}$$

$$-\nabla_c^2\varphi_1 - \frac{\lambda}{6}a^2\varphi_1 = \nabla_1\varphi_0 - \frac{\lambda}{6}\varphi_0^3. \tag{3.6}$$

(Higher order equations may be obtained in a straightforward manner if needed.) The set of differential equations obtained may be solved iteratively beginning with (3.5). By comparing (3.5) with (2.10) where ℓ_0^2 is defined by the critical length in (3.1), it is observed that φ_0 is the eigenfunction of the Laplacian whose eigenvalue is the smallest (with L set equal to L_c).

Because (3.5) is a homogeneous equation, the overall scale of the solution is not determined. However, (3.6), as well as the higher order equations not written down explicitly, are inhomogeneous. The scale of φ_0 can be fixed by requiring (3.4) to minimize the energy to lowest order in ϵ. From (2.6), using (3.4) along with (3.5) and (3.6), it follows that

$$E = \int_\Sigma \mathrm{d}\sigma_x \left\{\frac{\lambda}{24}a^4 - \frac{1}{2}\epsilon^2\varphi_0\nabla_1\varphi_0 + \frac{\lambda}{24}\epsilon^2\varphi_0^4\right\} \tag{3.7}$$

where we have kept only the lowest order terms in ϵ. If we let $\tilde{\varphi}_0$ be any solution to (3.5) which satisfies the correct boundary conditions, and write

$$\varphi_0 = A\tilde{\varphi}_0 \tag{3.8}$$

for some constant A, we obtain

$$E = C_0 - C_1A^2 + C_2A^4 \tag{3.9}$$

where

$$C_0 = \int_\Sigma \mathrm{d}\sigma_x \frac{\lambda}{24}a^4 \tag{3.10}$$

is independent of A and ϵ, and

$$C_1 = \frac{1}{2}\epsilon^2 \int_\Sigma \mathrm{d}\sigma_x \tilde{\varphi}_0\nabla_1\tilde{\varphi}_0 \tag{3.11}$$

$$C_2 = \frac{\lambda}{24}\epsilon^2 \int_\Sigma \mathrm{d}\sigma_x \tilde{\varphi}_0^4 \tag{3.12}$$

are independent of A. For $C_1 > 0$, we have

$$A = \pm\left(\frac{C_1}{2C_2}\right)^{1/2} \tag{3.13}$$

as the value of A which makes the energy a local minimum. This sets the scale of the solution for φ_0:

$$\varphi_0 = \pm\left(\frac{C_1}{2C_2}\right)^{1/2}\tilde{\varphi}_0 \tag{3.14}$$

with $\tilde{\varphi}_0$ any solution to (3.5).

To proceed to the next order in ϵ, we must solve (3.6) with φ_0 given by (3.14). The general solution may be expressed as

$$\varphi_1 = \varphi_{1h} + \varphi_{1p} \tag{3.15}$$

where φ_{1h} is a solution of the homogeneous equation (3.5) and φ_{1p} is any particular solution to (3.6). The overall scale of φ_{1p} is fixed since it satisfies an inhomogeneous equation; however, the scale of φ_{1h} is not determined. We can choose φ_{1h} to be proportional to $\tilde{\varphi}_0$ and fix the constant of proportionality by requiring that the energy be minimized to the next order in ϵ. We therefore take

$$\varphi(x) = \epsilon^{1/2}\left[\left(\frac{C_1}{2C_2}\right)^{1/2}\tilde{\varphi}_0 + \epsilon(A\tilde{\varphi}_0 + \varphi_{1p}) + \epsilon^2\varphi_2 + \cdots\right] \tag{3.16}$$

in (2.6) and minimize the resulting expression for $E(A)$. Because we are only interested in the value of A which makes $E(A)$ a minimum, it is simpler to evaluate

$$\frac{\partial}{\partial A}E(A) = \epsilon^{3/2}\int_\Sigma \mathrm{d}\sigma_x \tilde{\varphi}_0\left(-\nabla^2\varphi + \frac{\lambda}{6}\varphi^3 - \frac{\lambda}{6}a^2\varphi\right) \tag{3.17}$$

to lowest order in ϵ. Because we have only solved the equation of motion up to and including terms of order $\epsilon^{3/2}$ in obtaining (3.5) and (3.6), the next term in the integrand of (3.17) will be of order $\epsilon^{5/2}$. A short calculation shows that

$$\frac{\partial}{\partial A}E(A) = \epsilon^4(D_0 + D_1 A) \tag{3.18}$$

where

$$D_0 = \int_\Sigma \mathrm{d}\sigma_x\left[-\tilde{\varphi}_0\nabla_1\varphi_{1p} - \left(\frac{C_1}{2C_2}\right)^{1/2}\tilde{\varphi}_0\nabla_2\tilde{\varphi}_0 + \frac{\lambda}{2}\left(\frac{C_1}{2C_2}\right)\tilde{\varphi}_0^3\varphi_{1p}\right] \tag{3.19}$$

$$D_1 = \int_\Sigma \mathrm{d}\sigma_x\left[-\tilde{\varphi}_0\nabla_1\tilde{\varphi}_0 + \frac{\lambda}{2}\left(\frac{C_1}{2C_2}\right)\tilde{\varphi}_0^4\right]. \tag{3.20}$$

We conclude that if $D_1 > 0$, then

$$A = -\frac{D_0}{D_1} \tag{3.21}$$

gives the value which minimizes $E(A)$ to lowest order in ϵ. From (3.16), in summary, we therefore have the approximate solution given to order $\epsilon^{3/2}$ by

$$\varphi(x) = \epsilon^{1/2}\left[\left(\frac{C_1}{2C_2}\right)^{1/2}\tilde{\varphi}_0 + \epsilon\left(-\frac{D_0}{D_1}\tilde{\varphi}_0 + \varphi_{1p}\right)\right] \tag{3.22}$$

where $\tilde{\varphi}_0$ and φ_{1p} are any solutions to (3.5) and (3.6), respectively. C_1 and C_2 are defined by (3.11) and (3.12). D_0 and D_1 are defined by (3.19) and (3.20).

It should be clear from the analysis that we have presented how the method extends to any order in ϵ. First solve the equation of motion (2.3) to order $\epsilon^{n+1/2}$ using (3.4) extended to order $\epsilon^{n+1/2}$ and (3.3) to order ϵ^n. ($n = 0, 1, 2, \ldots$ here.) Evaluate $\frac{\partial}{\partial A}E(A)$ to order ϵ^{2n+2} using the solutions found, and then solve for A as we have illustrated for the cases $n = 0, 1$.

4. The rectangular cavity

4.1. Exact result

The simplest case of a cavity where an exact solution can be found occurs for a field confined inside of a rectangular box. The first case we will look at is for a field which satisfies a Dirichlet boundary condition on opposite sides of one pair of box walls. We will choose this to be the y direction and take $-L/2 \leqslant y \leqslant L/2$ with $\varphi = 0$ when $y = \pm L/2$. In the x and z directions, we will choose either periodic boundary conditions, or else Neumann boundary conditions with $\frac{\partial}{\partial x}\varphi$ and $\frac{\partial}{\partial z}\varphi$ vanishing at $x = \pm L_x/2$ and $z = \pm L_z/2$, respectively. With either of these two choices, the ground state will not depend on the x or z coordinates, and the problem reduces to one that is one-dimensional. One possible physical application of this is to the case of two parallel plates, as in the Casimir effect, where the plate separation is much less than their linear extent. (i.e., Keep L finite, and let $L_x, L_z \to \infty$.) In this case, with $L_x, L_z \to \infty$, the choice of boundary conditions in the x and z directions would not be expected to be important. Later in this section, we will discuss what happens in the case where Dirichlet boundary conditions are imposed in all three spatial directions. Of course, the present paper studies only a classical field theory, and the true Casimir effect has a quantum nature. To examine the role of the solutions found here in the Casimir effect, it would be necessary to study quantum fluctuations about the solutions. A similar problem was investigated in [4] but we will only concern ourselves with classical fields in the present paper. We only use the Casimir effect as a motivation for the boundary conditions adopted.

If we assume $\varphi = \varphi(y)$ only, then (2.5) becomes

$$-\frac{\mathrm{d}^2\varphi}{\mathrm{d}y^2} + \frac{\lambda}{6}\varphi(\varphi^2 - a^2) = 0. \tag{4.1}$$

This equation admits a first integral,

$$-\frac{1}{2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}y}\right)^2 + \frac{\lambda}{24}(\varphi^2 - a^2)^2 = C \tag{4.2}$$

with C a constant. Note that $\varphi = \pm a$ solves (4.1), but does not satisfy the requirement that the field vanishes at $\varphi = \pm L/2$, so is not allowed. $\varphi = 0$ satisfies (4.1) as well as the boundary conditions, so is a valid solution. To see if $\varphi = 0$ is locally stable we look for the solution to (2.10) of lowest eigenvalue. Because $\psi_0(y = \pm L/2) = 0$, we have $\psi_0 \propto \sin\left(\frac{\pi}{L}(y + L/2)\right)$ and $\ell_0^2 = \pi^2/L^2$. From (2.11), we see that $\varphi = 0$ is stable if $L < L_c$ where

$$L_c = \frac{\pi}{a}\left(\frac{6}{\lambda}\right)^{1/2}. \tag{4.3}$$

If $L > L_c$, then $\varphi = 0$ is unstable, and because the boundary conditions prohibit constant values of φ, the ground state must be spatially dependent. We will now find this solution.

For $\varphi(y)$ to be continuous on the interval $[-L/2, L/2]$ with $\varphi(\pm L/2) = 0$, it must have a stationary value somewhere. This allows us to conclude that C defined in (4.2) must be non-negative, and also that $C \leqslant \frac{\lambda}{24}a^4$. We will define

$$C = \frac{\lambda}{24}a^4 w^2 \tag{4.4}$$

with w real and satisfying $0 \leqslant w \leqslant 1$. The cases $w = 0$ and $w = 1$ require special treatment. For $w = 1$, it is easy to show that the only solution to (4.2) which satisfies the boundary conditions that $\varphi = 0$ at $y = \pm L/2$ is $\varphi = 0$ for all y. We already know that this solution is unstable for $L > L_c$, so we will concentrate on $w < 1$. The case $w = 0$ is simple to solve, and leads to the usual kink solution. (See [6, 7] for example.) This does not satisfy our boundary conditions. Therefore, we will restrict $0 < w < 1$. It is worth remarking on the difference between the confined case and the field in an unbounded region. In the latter case, the requirement that the field configuration leads to a finite energy requires the field to asymptotically approach a zero of the potential [6, 7]. For a field confined by a cavity, any well-behaved function will have a finite energy as a trivial consequence of the finite volume of the region; thus, the finite energy requirement does not give anything useful in our case.

If we restrict $0 < w < 1$, then the solution to (4.2) can be expressed in terms of Jacobi elliptic functions [8]. We find

$$\varphi(y) = \pm a(1-w)^{1/2}\mathrm{sn}\left(\beta\left(y+\frac{L}{2}\right), \left(\frac{1-w}{1+w}\right)^{1/2}\right) \tag{4.5}$$

where

$$\beta = \left[\frac{\lambda}{12}(1+w)a^2\right]^{1/2} \tag{4.6}$$

as the solution to (4.2) which vanishes at $y = -L/2$. If we impose the other boundary condition at $y = L/2$, we find the condition ($w \neq 1$)

$$\mathrm{sn}\left(\beta L, \left(\frac{1-w}{1+w}\right)^{1/2}\right) = 0. \tag{4.7}$$

Making use of the property [8] that $\mathrm{sn}(u, k)$ has zeros for $u = 2nK(k)$ for $n = 0, \pm 1, \pm 2, \ldots$ where $K(k)$ is the complete elliptic integral of the first kind, we conclude that

$$\beta L = 2nK\left(\left(\frac{1-w}{1+w}\right)^{1/2}\right) \tag{4.8}$$

where $n = 1, 2, \ldots$. We can restrict n to non-negative values since $\mathrm{sn}(u, k)$ is an odd function of u, and a sign change of u just changes the overall sign of φ which is arbitrary. We rule out $n = 0$ because this corresponds to φ identically zero in (4.5), and we already know that this is unstable for $L > L_c$. The interpretation of n is that $(n-1)$ gives the number of zeros of φ in the open interval $(-L/2, L/2)$.

The next question concerns the solution for the arbitrary integration constant w in (4.8). We need to solve

$$\gamma_n = (1+w)^{-1/2}K\left(\left(\frac{1-w}{1+w}\right)^{1/2}\right) \tag{4.9}$$

where

$$\gamma_n = \frac{La}{2n}\left(\frac{\lambda}{12}\right)^{1/2} \tag{4.10}$$

for w, with $n = 1, 2, \ldots$. The properties of the complete elliptic integral of the first kind may be used to show that the right-hand side of (4.9) is a monotonically decreasing function of w on the interval $0 \leqslant w \leqslant 1$, approaching infinity as $w \to 0$ and the value $\pi\sqrt{2}/4$ as $w \to 1$. This means that a solution to (4.9) only exists if

$$\gamma_n > \frac{\pi\sqrt{2}}{4}. \tag{4.11}$$

Making use of (4.10) and the definition of the critical length in (4.3) shows that we will always have a solution for w in (4.9) if

$$L > nL_c. \tag{4.12}$$

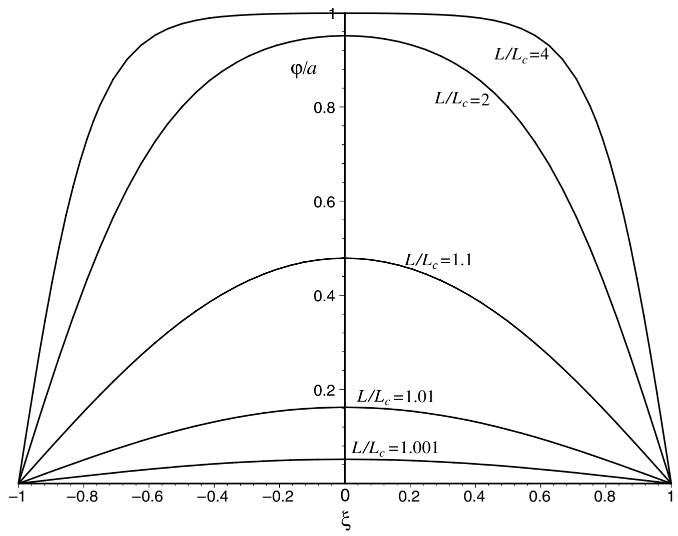

Figure 1. The solution for $\varphi(\xi)/a$ is shown for various values of L/L_c as a function of $\xi = 2y/L$.

Apart from the different boundary conditions, the situation is very similar to the twisted field case on the circle examined by Avis and Isham [3]. In fact, we can make use of their clever proof that for $n \geqslant 2$, the solutions for $\varphi(y)$ are all unstable. This is not unexpected, since the energy of solutions with $n \geqslant 2$ are all greater than that for the $n = 1$ solution. We will therefore concentrate on the case of $n = 1$.

If we use the definition of L_c given in (4.3) in (4.9), when $n = 1$, we find

$$\frac{\pi L}{2\sqrt{2}L_c}\sqrt{1+w} = K\left(\sqrt{\frac{1-w}{1+w}}\right) \tag{4.13}$$

showing that the solution for w depends only on the dimensionless ratio of L/L_c. By expanding both sides of (4.13) in powers of w, it is easy to show that for large values of L/L_c a good approximation for the solution for w is given by

$$w \simeq 8\exp\left(-\frac{\pi L}{\sqrt{2}L_c}\right). \tag{4.14}$$

Even for relatively small values of L/L_c, this turns out to be quite accurate. (For example, when $L/L_c = 3$, we find an agreement to six decimal places between this approximation and the result of solving (4.13) numerically.) The result for w in (4.13) can be shown to be a monotonically decreasing function of L/L_c, tending to zero as the value of L/L_c is increased, consistent with what is predicted from the approximation (4.14). Since $w = 0$ corresponds to the constant solution $\varphi = a$, it would be expected that as we increase the value of the ratio L/L_c, the solution for φ will try to be as close as it can to the constant value of a. This is in fact what happens as we show in figure 1. As L/L_c gets larger, the solution tends towards the step function, which is as close as the boundary conditions allow it to get to the standard solution $\varphi = a$.

4.2. Approximate result

As a test of the approximation method described in section 3, we will use it to compare with the exact solution we have just found. Take L to be close to the critical length L_c as in (3.2) with L_c defined by (4.3) in the present case. As explained in section 3, it is advantageous to adopt dimensionless coordinates, so we will define

$$y = \frac{L}{2}\xi \tag{4.15}$$

with $-1 \leqslant \xi \leqslant 1$. Since $\nabla^2 = \frac{d^2}{dy^2}$, by using (4.15) and the expansion of L about L_c as in (3.2), it is easy to read off ∇_1 and ∇_2.

The next step is to solve (3.5) for any solution $\tilde{\varphi}_0$. It is easy to see that

$$\tilde{\varphi}_0(\xi) = \cos\left(\frac{\pi}{2}\xi\right) \tag{4.16}$$

is a solution which satisfies the proper boundary conditions. A straightforward calculation of C_1 and C_2 defined by (3.11) and (3.12) gives the leading order approximation to the ground state as

$$\varphi(\xi) \simeq \left(\frac{8}{3}\epsilon\right)^{1/2} a\cos\left(\frac{\pi}{2}\xi\right). \tag{4.17}$$

To further demonstrate the approximation method, we will evaluate the next order correction to (4.17). This entails initially solving (3.6) for any solution φ_{1p}. With

$$\varphi_0(\xi) = \left(\frac{8}{3}\right)^{1/2} a\cos\left(\frac{\pi}{2}\xi\right) \tag{4.18}$$

it can be shown that a particular solution to (3.6) is given by

$$\varphi_{1p}(\xi) = -\frac{\sqrt{6}}{18}a\cos\left(\frac{3\pi}{2}\xi\right). \tag{4.19}$$

The constants D_0 and D_1 defined in (3.19) and (3.20) may now be evaluated with the result that the approximate ground state solution is

$$\varphi(\xi) \simeq \epsilon^{1/2}\left\{\left(\frac{8}{3}\right)^{1/2} a\cos\left(\frac{\pi}{2}\xi\right) - \epsilon\left[\frac{17\sqrt{6}}{36}a\cos\left(\frac{\pi}{2}\xi\right) + \frac{\sqrt{6}}{18}a\cos\left(\frac{3\pi}{2}\xi\right)\right]\right\} \tag{4.20}$$

up to, and including, terms of order $\epsilon^{3/2}$. (We drop the $\pm$ here.) This result can be shown to agree with an expansion of the exact solution in terms of elliptic functions found earlier.

The main conclusion of this section is that the approximation method is in complete agreement with the expansion of the exact result, at least for the first two orders in the expansion used. This is sufficient to generate some faith in the general procedure outlined in section 3 in cases where it is not possible to find an exact solution by analytical means. An example will be given in the following subsection. The approximation method we have described here can also be used to provide a useful check on the results of numerical calculations. We will study such a case in section 5.

4.3. Dirichlet boundary conditions in three dimensions

In this section, we will examine the case of a cubical box of side length L with the field vanishing on all of the box walls. Although this is perhaps a more realistic situation for a confined field than that considered in subsection 4.1, unfortunately we have not been able to find the exact solution when the zero field is unstable. This is quite unlike the situation where periodic or antiperiodic boundary conditions are imposed on the walls. In this case, the exact solution in three dimensions can be simply related to that found in one dimension [3]. The Dirichlet boundary conditions thwart the application of a similar procedure here. However, we can still use the approximation method described in section 3.

The stability of the solution $\varphi = 0$ is determined by the lowest eigenvalue ℓ_0^2 in (2.10). It is easy to show that

$$\psi_0(x, y, z) = \cos\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi y}{L}\right)\cos\left(\frac{\pi z}{L}\right) \tag{4.21}$$

is the relevant eigenfunction of the Laplacian with the lowest eigenvalue given by

$$\ell_0^2 = \frac{3\pi^2}{L^2}. \tag{4.22}$$

The critical length L_c is the value of L for which λ_0 defined in (2.11) vanishes. This gives

$$L_c^2 = \frac{18\pi^2}{\lambda a^2}. \tag{4.23}$$

For $L < L_c, \varphi = 0$ is stable, while for $L > L_c$, it is unstable. Thus, when $L > L_c$, the ground state will involve a non-constant value of φ in order to satisfy the Dirichlet boundary conditions.

It proves convenient to define dimensionless coordinates as in (4.15):

$$x = \frac{L}{2}\xi_1 \qquad y = \frac{L}{2}\xi_2 \qquad z = \frac{L}{2}\xi_3 \tag{4.24}$$

so that the boundary of the cube is at $\xi_i = \pm 1$ for $i = 1, 2, 3$. A solution to (3.5) with the correct boundary conditions is

$$\tilde{\varphi}_0(\xi_1, \xi_2, \xi_3) = \prod_{i=1}^{3} \cos\left(\frac{\pi}{2}\xi_i\right). \tag{4.25}$$

This may be used to calculate C_1 and C_2 in (3.11) and (3.12) resulting in the leading order approximation to the ground state, from (3.14), involving

$$\varphi_0(\xi_1, \xi_2, \xi_3) = \frac{8\sqrt{6}}{9} a \prod_{i=1}^{3} \cos\left(\frac{\pi}{2}\xi_i\right). \tag{4.26}$$

To proceed to the next order, we must solve the partial differential equation (3.6) with (4.26) substituted for φ_0. A solution with the correct boundary conditions can be shown to be

$$\begin{aligned}\varphi_{1p} = -\frac{2\sqrt{6}}{81} a \bigg\{ &\frac{1}{3}\cos\left(\frac{3\pi}{2}\xi_1\right)\cos\left(\frac{3\pi}{2}\xi_2\right)\cos\left(\frac{3\pi}{2}\xi_3\right) \\ &+ \frac{3}{2}\left[\cos\left(\frac{\pi}{2}\xi_1\right)\cos\left(\frac{3\pi}{2}\xi_2\right)\cos\left(\frac{3\pi}{2}\xi_3\right) + (1 \leftrightarrow 2) + (1 \leftrightarrow 3)\right] \\ &+ 9\left[\cos\left(\frac{\pi}{2}\xi_1\right)\cos\left(\frac{\pi}{2}\xi_2\right)\cos\left(\frac{3\pi}{2}\xi_3\right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)\right]\bigg\}\end{aligned} \tag{4.27}$$

where, to save space, we have used $(i \leftrightarrow j)$ to mean the first term in the square brackets with the indices i and j on ξ_i and ξ_j switched.

Finally, we evaluate D_0 and D_1 defined by (3.19) and (3.20). A straightforward calculation leads to

$$\frac{D_0}{D_1} = \frac{1375\sqrt{6}}{4374} a. \tag{4.28}$$

The solution may now be written down immediately from (3.22) with φ_0 given in (4.26) and φ_{1p} in (4.27). (To save space we will not write this out explicitly.)

5. The spherical cavity

We now examine the case where the scalar field is confined by a spherical shell of radius R, with the field vanishing at $r = R$. The ground state should be spherically symmetric, so that φ will only depend on the radial coordinate r if we use the usual spherical polar coordinates. We need to solve

$$-\nabla_r^2\varphi + \frac{\lambda}{6}\varphi(\varphi^2 - a^2) = 0 \tag{5.1}$$

where $\varphi = \varphi(r)$ and

$$\nabla_r^2 = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\right) \tag{5.2}$$

with $\varphi(r = R) = 0$.

$\varphi = 0$ is obviously a valid solution, and from the discussion in section 2, the stability is determined once we know the lowest eigenvalue of the Laplacian as in (2.10). In the present situation, it is easy to show that

$$\psi_0 = \frac{\sin(\pi r/R)}{r} \tag{5.3}$$

is the eigenfunction of lowest eigenvalue, with

$$\ell_0^2 = \left(\frac{\pi}{R}\right)^2. \tag{5.4}$$

From (2.11), we can conclude that $\varphi = 0$ is stable for $R < R_c$ where the critical shell radius R_c is defined by

$$R_c = \left(\frac{6}{\lambda}\right)^{1/2}\frac{\pi}{a}. \tag{5.5}$$

For $R > R_c$, $\varphi = 0$ is not the ground state.

To find the stable ground state when $R > R_c$, we must solve the non-linear differential equation (5.1). We were unable to find an exact analytical solution here, unlike the case of the one-dimensional box where the differential equation was able to be solved by quadrature. Instead we solved the equation by numerical integration. The results are plotted in figure 2 for a range of values of R/R_c. The results show that as the value of R/R_c is increased, the value of the field tries to get closer and closer to the constant value of $\varphi = a$ over a greater range of values of the radius. Because the boundary conditions require $\varphi(r = R) = 0$, this results in a very sharp drop-off in the value of the field as $r \to R$. The profile of the field starts to approach a step-function as $R/R_c \to \infty$.

Although we were not able to find an analytic solution to (5.1), we can still apply the approximation method described in section 3. A dimensionless radial coordinate ρ is defined by $r = R\rho$, and we set $R = (1+\epsilon)R_c$ as in (3.2). The lowest order contribution to the approximate solution follows from solving (3.5) and is proportional to the lowest eigenfunction (5.3). The overall scale is set by calculating C_1 and C_2 in (3.11) and (3.12) with the result that

$$\varphi_0(\rho) = \frac{a\sin(\pi\rho)}{\sqrt{I}\rho} \tag{5.6}$$

with

$$I = \frac{\pi}{2}[\mathrm{Si}(2\pi) - \mathrm{Si}(4\pi)]. \tag{5.7}$$

Here $\mathrm{Si}(x)$ denotes the Sine integral.

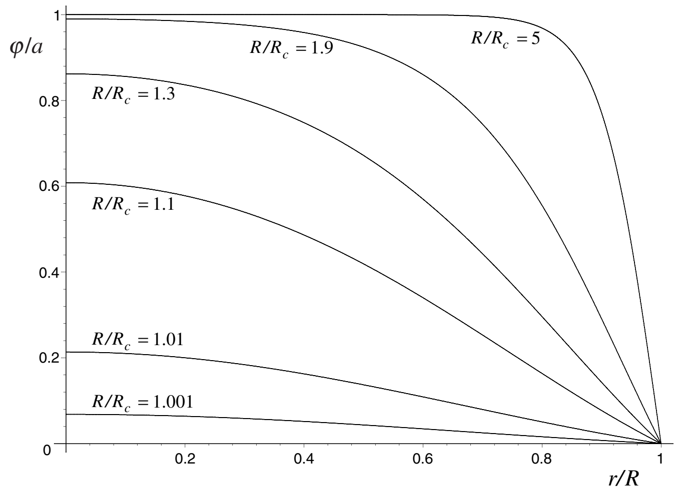

Figure 2. The solution for $\varphi(r)/a$ is shown for various values of R/R_c as a function of r/R.

Table 1. A table showing the value of the field at the origin in units of a. ϵ shows how close R is to R_c as defined by $R = (1+\epsilon)R_c$. The second column gives the true value of $\varphi(0)/a$ found by numerical integration of (5.1). The final two columns show respectively the results found from the approximate solution (3.22) using the leading order term, and the next order correction.

ϵ	$\varphi(0)/a$	Order $\epsilon^{1/2}$	Order $\epsilon^{3/2}$
0.001	0.068 291	0.068 370	0.068 288
0.01	0.213 584	0.216 206	0.213 590
0.1	0.608 397	0.683 703	0.601 004
0.2	0.774 283	0.966 902	0.732 994
0.3	0.862 136	1.184 208	0.754 491

A particular solution to (4.6) can be shown to be

$$\varphi_{1p}(\rho) = -\frac{\pi^2 a}{2I^{3/2}}[\mathrm{Ci}(4\pi\rho) - \mathrm{Ci}(2\pi\rho)]\frac{\sin(\pi\rho)}{\rho} + \frac{\pi^2 a}{2I^{3/2}}[\mathrm{Si}(4\pi\rho) - 2\mathrm{Si}(2\pi\rho)]\frac{\cos(\pi\rho)}{\rho} + \frac{\pi a}{I^{1/2}}\cos(\pi\rho). \tag{5.8}$$

Here Ci the Cosine integral. The complicated nature of this renders the calculation of D_0 in (3.19) difficult, although D_1 in (3.20) is easily evaluated. It can be shown that

$$\frac{D_0}{D_1} \simeq 0.417\,349\,770a. \tag{5.9}$$

This is sufficient to determine the approximate solution in (3.22) correct to order $\epsilon^{3/2}$.

As a check on the numerical results shown in figure 2, we plotted the approximate solution we have just described. For small ϵ, the result was found to be indistinguishable from the result of numerical integration of (5.1); however, as ϵ is increased the agreement becomes less good as would be expected for a small ϵ expansion. The disagreement is largest near $\rho = 0$ where the field has its largest value. To get an idea of how close the approximate solution is to the true result, some of the values found for the field at the origin are included in table 1. It is clear that by including the correction to the leading term a more accurate result is obtained for a wider range of ϵ. Even for relatively large values of ϵ ($\epsilon \simeq 1$) where the small ϵ expansion would not be expected to be particularly accurate, we found the approximate solution to be close to the true one for ρ close to $\rho = 1$, corresponding to the radius of the shell. Thus, the approximation method described in section 3 can provide a useful check on the results of numerical integration in cases where no exact solution is known.

6. Summary and conclusions

We have presented an analysis of the ground state for a real scalar field confined in a cavity. Although we concentrated on the double-well potential (2.2), it would be easy to extend the analysis to other potentials. (For example, in the case the bi-cubic potential, an exact solution in the cavity can be found in terms of the Weierstrass elliptic function as in [10].) In addition, the analysis can be extended to curved space, although it would be difficult to find exact solutions except in special cases.

We presented a method for obtaining approximate analytical solutions in the case where the length scales of the cavity were close to the critical values at which $\varphi = 0$ became unstable. For the rectangular cavity, we showed that this approximation method agreed with an expansion of the exact solution which we found. When $\varphi = 0$ was not the ground state, it was shown that as the ratio of the size of the cavity to the critical size L/L_c increases, the ground state tends towards the constant value which minimizes the potential over as much of the cavity as possible. The condition that the field vanishes on the box walls means that the field must always drop to zero, with the drop-off becoming increasingly sharp as L/L_c is increased. A similar behaviour was found for the case of a spherical cavity. By extrapolation, for a field confined to vanish on the walls of a general cavity, if the cavity is sufficiently large it would be expected that the ground state would correspond to a field which was constant almost everywhere inside the cavity, with a very sharp drop-off to zero as the cavity boundary is approached.

Although we have concentrated on fields confined by cavities in the present paper, the general method described above can be useful in other situations where the boundary conditions prohibit constant values of the field. We plan to report on this elsewhere.

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